Non-Laplacian growth, algebraic domains and finite reflection groups

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Abstract

Dynamics of planar domains with moving boundaries driven by the gradient of a scalar field that satisfies an elliptic PDE is studied. We consider the question: For which kind of PDEs the domains are algebraic, provided the field has singularities at a fixed point inside the domain? The construction reveals a direct connection with the theory of the Calogero-Moser systems related to finite reflection groups and their integrable deformations.

1 Introduction, non-homogeneous porous medium flows

Laplacian growth is a process that governs the dynamics of the boundary $\partial\Omega = \partial\Omega(t)$ in the plane separating two disjoint, open regions (Ω and $\mathbb{C}\backslash\bar{\Omega}$) in which harmonic (scalar) fields are defined. These may be interpreted as the pressure fields for incompressible fluids (porous-medium flows, Hele-Shaw flows, etc. See e.g. [11] and references therein). In the recent literature there have appeared several new formulations of the Laplacian growth related with the theory of integrable systems and random matrices, quantum Hall effect and Dirichlet boundary problem in two-dimensions (see e.g. [6], [7], [12] and references therein).

In the present paper, we consider a new connection with the theory of quantum integrable systems. To be more precise, we study an integrable generalization of the Laplacian growth, when the boundary is driven by a field satisfying an elliptic PDE, that is not generally reduced to a Beltrami-Laplace equation ("Non-Laplacian" growth). To be specific, we use the porous medium fluid dynamics interpretation.

We find variable-coefficient elliptic PDEs for which the boundary dynamics can be described explicitly and the moving fluid occupies evolving algebraic domains (see below). These, turn out, to be PDEs of the Calogero-Moser type, related to finite reflection (Coxeter) groups as well as their integrable deformations (see Conclusion), that possibly complete the list of all second-order PDEs

connected with the algebraic domains (our main conjecture).

In this section we set the problem in terms of the porous medium fluid dynamics. Formulation of the problem in terms of quadrature domains is given in the next section.

Consider a flow of a liquid in a thin non-planar layer of non-homogeneous porous medium. The layer can be viewed as a 2-dimensional surface embedded in the three-dimensional Euclidean space. We let the layer curvature, permeability, porosity and thickness depend on the surface spatial coordinates x, y. We can choose x, y such that locally

$$dl^2 = G(x,y)(dx^2 + dy^2)$$

where dl is the surface length element. The surface area element is $d\Sigma = Gdxdy$, and the volume of the liquid that can be absorbed in the range $x + dx, y + dy$ equals

$$dV = \eta h d\Sigma = \eta h G dxdy,$$

where $\eta = \eta(x,y), h = h(x,y)$ are the medium porosity and the layer thickness, respectively.

In the porous medium, the flow velocity $v = (dx/dt, dy/dt)$ is proportional to the gradient $\nabla = (\partial/\partial x, \partial/\partial y)$ of the pressure P

$$v = -\frac{\kappa}{\sqrt{G}}\nabla P,$$

where $\kappa = \kappa(x,y)$ is the medium permeability.

It is seen from the above that only the two combinations of variable coefficients, namely

$$\eta h G, \quad \frac{\kappa}{\sqrt{G}}$$

enter the flow equation of motion, and it is convenient to absorb h and G into definitions of the other coefficients. Therefore, without loss of generality, we can consider the flow in the plane parametrized by the complex coordinates $z = x + iy, \bar{z} = x - iy$, choosing η and κ to depend on $z, \bar{z}$, while setting remaining coefficients to unity. The liquid volume conservation leads to the continuity equation

$$(\nabla \cdot \eta v) = 0, \tag{1}$$

while the dynamical law of motion rewrites as

$$v = -\kappa \nabla P. \tag{2}$$

We consider a situation where the liquid occupies a bounded, simply-connected open region Ω of the plane, whose time evolution $\Omega = \Omega(t)$ is induced by the flow.

At fixed time t, the pressure is constant along the boundary

$$P(\partial\Omega(t)) = P_0(t). \tag{3}$$

Note, that the in the case of the simply-connected domains considered here, the dynamics is independent of $P_0(t)$, and for convenience the latter can be set to zero.

The normal velocity of the boundary v_n and that of the flow coincide at $\partial\Omega$

$$v_n = n \cdot v \quad \text{if} \quad z \in \partial\Omega. \tag{4}$$

The flow is singularity driven. For instance

$$P \to \frac{-q(t)}{2\kappa(z_1,\bar{z}_1)\eta(z_1,\bar{z}_1)} \log|z-z_1| + \sum_{j=1}^{k} \left(\frac{\mu_j(t)}{(z-z_1)^j} + \frac{\bar{\mu}_j(t)}{(\bar{z}-\bar{z}_1)^j} \right), \quad \text{as} \quad z \to z_1, \tag{5}$$

when a multipole source of order $k+1$ is located at $z = z_1$. Equations (1) - (5) constitute a free boundary problem where the evolution of the boundary $\partial\Omega(t)$ is completely determined by the initial condition $\partial\Omega(0)$ and strengths $q(t) = \bar{q}(t), \mu_j(t), \bar{\mu}_j(t), j = 1, .., k$ as well as position z_1 of the sources.

2 Conservation Laws and Quadrature Domains

From (1), (2), (5) it follows that the pressure satisfies the elliptic PDE

$$\nabla\kappa\eta\nabla P = -\pi\hat{q}[\delta(x-x_1)\delta(y-y_1)]. \tag{6}$$

where $\hat{q} = \hat{q}(t)$ is the differential operator of order k

$$\hat{q} = q(t) + \sum_{j=1}^{k} (-1)^j \left(q_j(t) \frac{\partial^j}{\partial z^j} + \bar{q}_j(t) \frac{\partial^j}{\partial \bar{z}^j} \right), \quad \bar{q} = q \tag{7}$$

Let $\phi(z,\bar{z})$ be a time-independent function satisfying

$$\nabla\kappa\eta\nabla\phi = 0, \quad z \in \Omega \tag{8}$$

in whole Ω, including $z = z_1$.

Let us now estimate the time derivatives of the following quantities

$$M[\phi] = \int_{\Omega(t)} \eta\phi dxdy.$$

Considering an infinitesimal variation of the fluid domain $\Omega(t) \to \Omega(t+dt)$, we get

$$\frac{dM[\phi]}{dt} = \oint_{\partial\Omega(t)} v_n \eta\phi dl,$$

where dl is the boundary arc length. From (2), (3), (4) it follows

$$\frac{dM[\phi]}{dt} = \oint_{\partial\Omega(t)} (P\kappa\eta\nabla\phi - \phi\kappa\eta\nabla P)\cdot n dl.$$

Applying the Stokes theorem and remembering that P and ϕ satisfy (6), (8), we get

$$\frac{dM[\phi]}{dt} = \pi\hat{q}(t)[\phi](z_1, \bar{z}_1). \tag{9}$$

Note that mixed derivatives are absent $\hat{q}$ (c.f. (7)), for by (8), $\frac{\partial^2\phi}{\partial z\partial\bar{z}}$ is expressed through first derivatives of ϕ. $\bar{q}_j$ is the complex conjugate of q_j, since both $\phi(z,\bar{z})$ and $\bar{\phi}(\bar{z}, z)$ satisfy (8).

It follows that $M[\phi]$ is conserved for any solution of (8), a such that $\hat{q}[\phi](z_1, \bar{z}_1) = 0$.

The conservation laws have been first obtained for the homogeneous medium flows in [9], the variable-coefficient generalization seems to be first presented in [5].

The flow in the homogeneous medium

$$\kappa = 1, \quad \eta = 1 \tag{10}$$

is the simplest example, where the conservation laws can be written down explicitly [9]. In this example, any (anti)analytic in Ω function satisfies (8)

$$\phi(z,\bar{z}) = f(z) + g(\bar{z}), \quad \text{for} \quad \kappa = 1, \quad \eta = 1, \tag{11}$$

where f, g are univalent in Ω and the quantities

$$\int_{\Omega(t)} \left(f(z)(z - z_1)^k + g(\bar{z})(\bar{z} - \bar{z}_1)^k\right) dxdy$$

are integrals of motion for the free-boundary flows driven by a multipole source of order $k+1$ located at $z = z_1$ in homogeneous medium.

Returning to the general case, we integrate (9) getting

$$M[\phi](t) = M[\phi](0) + \pi\hat{Q}[\phi](z_1, \bar{z}_1),$$

where

$$\hat{Q} = \int_0^t \hat{q}(t')dt' = Q + \sum_{j=1}^{k}\left(Q_j\frac{\partial^j}{\partial z^j} + \bar{Q}_j\frac{\partial^j}{\partial\bar{z}^j}\right), \quad . \tag{12}$$

Therefore $M[\phi](t)$, and consequently a form of the domain, does not depend on the history of the sources and is a function of "multipole fluxes"

$$Q = \int_0^t q(t')dt',, \quad Q_j = \int_0^t q_j(t')dt', \quad \bar{Q}_j = \int_0^t \bar{q}_j(t')dt', \quad j = 1..k$$

injected by time t.

Now consider the special case when $M[\phi](0) = 0$ that describes the injection of the fluid to an initially empty medium. In such a case

$$\int_{\Omega} \eta(z, \bar{z})\phi(z, \bar{z})dxdy = \pi\hat{Q}[\phi(z_k, \bar{z}_k)], \tag{13}$$

Equation (13) is a generalization of quadrature identities (those expressing integrals over Ω through evaluation of integrands and a finite number of their derivatives at a finite number of points inside Ω) appearing in the theory of harmonic functions [10] to the case of elliptic equations with variable coefficients. Special domains for which the quadrature identities hold are called quadrature domains in the theory of the harmonic functions. We extend this definition to solutions of an arbitrary elliptic PDE with regular in Ω coefficients.

To construct quadrature domains (or equivalently the domains formed by fluid injected into initially empty porous medium), one needs an explicit form of a general solution of (8). Such explicit solutions are available for a class of the second order differential equations that are related to the Schrodinger operators of integrable systems on the plane. We, however, postpone construction of these solution to Section 4 and first present our main result in the next section.

3 The Main Result

Let us start with the simplest possible example, where the liquid is injected in the initially empty homogeneous porous medium through the single monopole source at $z = z_1 = x_1 + iy_1$. By symmetry of the problem the solution is a circular disc of the radius $r(t)$, centered at $z = z_1$

$$|z - z_1| < r(t). \tag{14}$$

The pressure satisfies

$$\Delta P = -\pi\tilde{q}(t)\delta(x - x_1)\delta(y - y_1),$$

where the source power and the total flux are

$$\tilde{q}(t) = \frac{dr(t)^2}{dt}, \quad \tilde{Q} = r^2,$$

respectively.

The remarkable fact is that the variable-coefficient problem, with the constant porosity and the permeability that varies as an inverse square of one Cartesian coordinate

$$\kappa = \frac{1}{x^2}, \quad \eta = 1 \tag{15}$$

admits the same circular solution (14) if the flow is driven by a combination of the same monopole source (of strength $q = \tilde{q}(t)$) and a dipole source of strength $q_1 = -\tilde{q}\tilde{Q}/2x_1$, both located at the point $z = z_1$

$$\nabla \frac{1}{x^2} \nabla P = -\pi \hat{q}[\delta(x - x_1)\delta(y - y_1)], \quad \hat{q} = \frac{dr^2}{dt}\left(1 - \frac{r^2}{2x_1}\frac{\partial}{\partial x}\right). \tag{16}$$

Indeed, it is not difficult to check that

$$P = r\frac{dr}{dt}\left((2x_1 x + \rho^2 + r^2)\log\rho - \frac{r^2 x(x - x_1)}{\rho^2} - \rho^2 + x(x - x_1) - (2x_1 x + \rho^2)\log(r)\right), \tag{17}$$

where $\rho = |z - z_1|$, satisfies (16), and the boundary conditions (2), (15),

$$\frac{dr}{dt} = -\frac{1}{x^2}\left(\frac{\partial P}{\partial n}\right)_{\rho = r}$$

as well as (6) holds at the disc boundary.

Also the following quadrature identity holds

$$\int_{|z - z_1| < r} \phi dx dy = \pi r^2 \phi(z_1, \bar{z}_1) + \frac{\pi r^4}{4x_1}\left(\frac{\partial \phi}{\partial x}\right)_{z = z_1},$$

which is a simplest non-trivial generalization of the mean value theorem for harmonic functions to the case of regular in (14) solutions of the elliptic PDE

$$\nabla \frac{1}{x^2} \nabla \phi = 0.$$

The above example is a special case of the main result presented below.

Let $\Omega(t)$ be a domain, resulted from the injection of the fluxes $\tilde{Q}, \tilde{Q}_j$, $j = 1..\tilde{k}$ into a **homogeneous** *medium through a multipole source of order $\tilde{k}+1$ located at $z = z_1$. Then the same domain can be formed by an injection of a special combination of fluxes Q, Q_j, $j = 1..k$ through a multipole source of order $k = (\tilde{k}+1)(s(n+l)+1)$ located at $z = z_1$ into an initially empty* **nonhomogeneous** *medium with permeability*

$$\kappa = \frac{1}{(z^s + \bar{z}^s)^{2n}(z^s - \bar{z}^s)^{2l}}, \quad n > l \geq 0, \quad s > 0, \quad \eta = 1 \tag{18}$$

and constant porosity, where s, n, l are integers.

In more details, the multipole fluxes of nonhomogeneous medium problem must be fixed functions of fluxes of its homogeneous medium counterpart

$$Q = \tilde{Q}, \quad Q_j = Q_j(\tilde{Q}, \tilde{Q}_1, ..., \tilde{Q}_{\tilde{k}}, \bar{\tilde{Q}}_1, ..., \bar{\tilde{Q}}_{\tilde{k}}, z_1, \bar{z}_1), \quad j = 1..(\tilde{k}+1)(s(n+l)+1) - 1.$$

For instance, in the above example of circular $\tilde{k} = 0$ solution in a medium with permeability (15) (that is the special case $n = 1, l = 0, s = 1$ of (18))

$$k = (0 + 1)(1(1 + 0) + 1) - 1 = 1, \quad Q = \tilde{Q}, \quad Q_1 = \frac{\tilde{Q}^2}{4x_1}.$$

Note that (18) can be rewritten in the form

$$\kappa = \frac{1}{\zeta(x, y)^2}, \quad \zeta(x, y) = \prod_{\alpha \in \mathcal{R}_+} (\alpha \cdot z)^{m_\alpha}, \quad (a \cdot b) := \mathrm{Re}(\bar{a}b),$$

where $\mathcal{R} = \{\alpha\}$ is a set of root vectors of a finite reflection (Coxeter) group on the plane (Dihedral group). $\mathcal{R}$ is invariant under reflections $z \to z - 2\frac{(\alpha \cdot z)}{(\alpha \cdot \alpha)}\alpha$ in a mirror (line), normal to any root vector $\alpha \in \mathcal{R}$, and $\mathcal{R}_+$ denotes a positive subset of $\mathcal{R}$, containing a half of all root vectors. The multiplicities m_α must be non-negative integers that are functions on the group orbits. If $l = 0$, the reflection group has one orbit and $m_\alpha = n$. Otherwise, the group has two orbits and multiplicities m_α take values n and l on each orbit respectively. The permeability (18) is therefore an invariant of the group of symmetries of a regular $4s$-polygon ($2s$-polygon if $l = 0$) and its singular locus coincides with the union of mirrors.

The pressure satisfies the elliptic PDE

$$\nabla \zeta(x, y)^{-2} \nabla P = \zeta(x, y)^{-1} H \zeta(x, y)^{-1} P = -\pi \hat{q}[\delta(x - x_k)\delta(y - y_k)]$$

$$\mathrm{order}(\hat{q}) = (\tilde{k} + 1)\,(\deg(\zeta(x, y)) + 1) - 1 = (\tilde{k} + 1)\left(1 + \sum_{\alpha \in \mathcal{R}_+} m_\alpha\right) - 1,$$

where H is the Schrödinger operator of the Calogero-Moser system related to the dihedral group [8], [4], [2]

$$H = \Delta - \sum_{\alpha \in \mathcal{R}_+} (\alpha \cdot \alpha) \frac{m_\alpha(m_\alpha + 1)}{(\alpha \cdot z)^2}.$$

Also, regular in Ω solutions $\phi(z, \bar{z})$ of the elliptic PDEs $\nabla \zeta(x, y)^{-2} \nabla \phi = 0$ satisfy the quadrature identities

$$\int_\Omega \phi dx dy = \hat{Q}[\phi](z_1, \bar{z}_1), \quad \mathrm{order}(\hat{Q}) = (\tilde{k} + 1)\,(\deg(\zeta(x, y)) + 1) - 1$$

that are generalization of the mean value theorem for harmonic functions to the case of solutions of the variable-coefficient elliptic PDEs in (generally) non-circular domains.

Before proving the main result we need to construct a complete set of solutions to (8) for the medium with κ, η given by (18).

4 Nonhomogeneous porous medium flows and integrable systems related to the finite reflection groups

In this section we show how to obtain a general solution $\phi(z, \bar{z})$ to (8), (18). It is instructive to consider the special case $s = 1, l = 0$ of (18)

$$\kappa = \frac{1}{x^{2n}}, \quad \eta = 1 \tag{19}$$

when permeability depends on one Cartesian coordinate only. Construction of solutions in the general case (18) is conceptually similar.

Consider the simplest non-trivial example $n = 1$ in (19) and start with factorizing the differential operator ∂_x^2 as

$$\frac{\partial^2}{\partial x^2} = \left(\frac{1}{x}\frac{\partial}{\partial x}\right)\left(x\frac{\partial}{\partial x} - 1\right).$$

By the associativity of the differential operators

$$(x\tfrac{\partial}{\partial x} - 1) \quad \underbrace{\left(\frac{1}{x}\frac{\partial}{\partial x}\right)\left(x\frac{\partial}{\partial x} - 1\right)}_{\partial_x^2} \quad = \quad \underbrace{\left(x\frac{\partial}{\partial x} - 1\right)\left(\frac{1}{x}\frac{\partial}{\partial x}\right)}_{x^2\partial_x\frac{1}{x^2}\partial_x} \quad (x\tfrac{\partial}{\partial x} - 1) \qquad .$$

Therefore,

$$T_1\Delta = L_1 T_1, \quad T_1 = x\frac{\partial}{\partial x} - 1, \quad L_1 = x^2\nabla\frac{1}{x^2}\nabla \tag{20}$$

and the elliptic equation (8) is nontrivially related to the Laplace equation when $\kappa\eta = 1/x^2$. In general, the identity

$$T\Delta = LT$$

that relates two differential operators (e.g. Δ and L above) through a differential operator T is called an intertwining identity and T is an intertwining operator.

A differential operator that is related to Δ through the intertwining identity equals, modulo a gauge transformation, a Schrödinger operators of an integrable system. Indeed,

$$T\,(\Delta - \lambda) = (L - \lambda)\,T$$

and when ν is an eigenfunction of Δ with the eigenvalue λ, $T[\nu]$ is an eigenfunction of L with the same eigenvalue or zero.

The factorization approach leading to the simplest nontrivial intertwining identity (20) can be now applied to L_1 etc. By induction we get the intertwining identity for an arbitrary nonnegative integer n in (19)

$$T_n\Delta = L_n T_n, \quad L_n = x^{2n}\nabla\frac{1}{x^{2n}}\nabla, \tag{21}$$

where

$$T_n = x^n \left(\frac{\partial}{\partial x} - \frac{n}{x}\right)\left(\frac{\partial}{\partial x} - \frac{n-1}{x}\right)...\left(\frac{\partial}{\partial x} - \frac{1}{x}\right) = \sum_{i=0}^{n} a_{i;n} x^i \frac{\partial^i}{\partial x^i}. \qquad (22)$$

Any solution ϕ to (8), (19) in Ω can be represented in the form

$$\phi = T_n[f], \quad \Delta f = 0, \quad z \in \Omega. \qquad (23)$$

Let us show this for $n = 1$, where T_1 is given by (20). Introduce f satisfying the following equation

$$x\frac{\partial f}{\partial x} - f = \phi,$$

where $L_1[\phi] = 0$ in Ω and $\phi = 0$ for $z \notin \Omega$. It is not difficult to see that

$$f(z, \bar{z}) = x \int_{-\infty}^{x} \frac{\phi(x' + iy, x' - iy)}{(x')^2} dx' + xF(y),$$

where $F(y)$ is an arbitrary regular function of y. Therefore, for any ϕ regular in Ω (which is the case) there exist a regular in Ω function f, such that $T_1[f] = \phi$. From the intertwining identity (20) it follows that $T_1\Delta f = 0$ and $\Delta f \in \mathrm{Ker}(T_1)$, if $z \in \Omega$ i.e.

$$\Delta[f] = xC(y), \quad z = x + iy \in \Omega$$

where $C(y)$ is an arbitrary function of y. It follows that

$$\Delta[f + xF(y)] = 0, \quad \frac{d^2F(y)}{dy^2} = C(y).$$

Since f is defined modulo $xF(y)$, we can set $F = 0$ and any solution of $L_1[\phi] = 0$ can be represented as $\phi = T_1[f]$, where $\Delta[f] = 0$. Similar proof applies to the arbitrary n case (23).

In the general case (18)

$$T_{n,l;s}\Delta = L_{n,l;s}T_{n,l;s}, \quad L_{n,l;s} = (z^s + \bar{z}^s)^{2n}(z^s - \bar{z}^s)^{2l}\nabla \frac{1}{(z^s + \bar{z}^s)^{2n}(z^s - \bar{z}^s)^{2l}}\nabla,$$

where the intertwining operator can be expressed in the form of a Wronskian [2]

$$T_{n,l;s}[f] = \rho^{s(n+l)} \frac{W[\sin(\theta_1), \sin(\theta_2), ..., \sin(\theta_n), f]}{\cos(s\theta)^{n(n-1)/2}\sin(s\theta)^{l(l-1)/2}}, \quad W[f_1, .., f_k] := \det\left[\frac{\partial^{j-1} f_i}{\partial \theta^{j-1}}\right]_{1 \le i,j \le k} \qquad (24)$$

with

$$z = \rho e^{i\theta}, \quad \theta_k = \left\{ \begin{array}{cc} k\left(s\theta + \frac{\pi}{2}\right), & k = 1, 2, ..n - l \\ (2k + l - n)\left(s\theta + \frac{\pi}{2}\right), & n - l < k \le n \end{array} \right. .$$

It is important that the intertwining operator $T_{n,l;s}$ is a homogeneous differential polynomial in $z, \bar{z}, \partial_z, \partial_{\bar{z}}$. This fact allows one to construct the quadrature domains for solutions ϕ of $L_{n,l;s}[\phi] = 0$.

Note that in general, there exist several independent operators intertwining Δ and another second order differential operator (forming a linear space of intertwining operators). For instance, T_n in (21) that intertwines Δ with $L_n = L_{n,0;1}$ is not a special case $T_{n,0;1}$ of (24). However, any nonzero linear combination of them can be used to obtain solutions to the corresponding elliptic equations $L_{n,0;1}[\phi] = 0$.

5 Proof of the main result

The zero-initial condition solution of the homogeneous-porous medium flow that is driven by a multipole source of order $\tilde{k} + 1$ is described by the polynomial conformal map of degree $\tilde{k} + 1$ from the unit disc in the parametric $|w|$-plane into the fluid region Ω

$$z(w) = z_1 + rw + \sum_{i=1}^{\tilde{k}} u_i w^{i+1}, \quad |w| < 1 \tag{25}$$

The map is analytic in $|w| < 1$, and the unit circle $|w| = 1$ is mapped to the boundary $\partial\Omega$. As shown above such regions are also quadrature domains. They are special (polynomial) cases of (rational) algebraic domains [11]. The map coefficients $r, u_i, i = 1, ..\tilde{k}$ are functions of $\tilde{Q}, \tilde{Q}_i, \bar{\tilde{Q}}_i, i = 1..\tilde{k}$. The "conformal radius" r can be chosen to be real.

The main idea of the proof is to show that the quadrature identity (13) for solutions of (8), (18) holds in domains defined by (25).

Consider illustrative examples of the problem (19) with permeability changing in one direction. As shown in the previous Section any solution of the elliptic equation $L_n[\phi] = 0$ can be represented as

$$T_n[f(z) + g(\bar{z})], \tag{26}$$

where $f(z), g(\bar{z})$ are holomorphic and anti-holomorphic respectively.

According to (13) we have to show that

$$\int_\Omega T_n[f(z)]dxdy = \pi\hat{Q}T_n[f(z)]_{z=z_1}$$

holds for any analytic in Ω function $f(z)$ when Ω is defined by the conformal map (25). Using the Green theorem and taking (22) into account we rewrite the last equation as

$$\sum_{j=0}^{n} \frac{a_{j;n}}{2^j(j+1)} \frac{1}{2\pi i} \oint_{\partial\Omega} (z + \bar{z})^{j+1} \frac{\partial^j f(z)}{\partial z^j} dz = \hat{Q}T_n[f(z)]_{z=z_1}. \tag{27}$$

Since $\bar{w} = 1/w$ if $|w| = 1$, $\bar{z}(\bar{w}) = \bar{z}(1/w)$ along the boundary, and we can rewrite the left-hand side of the last equation as

$$\sum_{j=0}^{n} \frac{a_{j;n}}{2^j(j+1)} \frac{1}{2\pi i} \oint_{|w|=1} \left((z(w) + \bar{z}(1/w))^{j+1} \left(\frac{1}{\frac{\partial z(w)}{\partial w}} \frac{\partial}{\partial w} \right)^j [f(z(w))] \right) \frac{\partial z(w)}{\partial w} dw.$$

Since $z(w)$ is analytic in $|w| < 1$, $\bar{z}(1/w)$ has poles only at $w = 0$, and the above integral is a pure sum of residues

$$\sum_{j=0}^{(\tilde{k}+2)(n+1)-2} V_j \left(\frac{\partial^j f(z)}{\partial z^j} \right)_{z=z_1}, \tag{28}$$

where $V_j, j = 0..(\tilde{k}+2)(n+1)-2$ are functions of the parameters $z_1, \bar{z}_1, r, u_j, \bar{u}_j, j = 1..\tilde{k}$ of the conformal map (25). Equating it with the right-hand side of (27), we see that $\hat{Q}_k$ must be (differential operators) of order $(\tilde{k}+1)(n+1) - 1$ and

$$\hat{Q}T_n[f(z)]_{z=z_1} = \sum_{j=0}^{(\tilde{k}+2)(n+1)-2} U_j \left(\frac{\partial^j f(z)}{\partial z^j} \right)_{z=z_1}, \tag{29}$$

where $U_j, j = 0..(\tilde{k}+2)(n+1) - 2$ are linear functions of $Q, Q_j, \bar{Q}_j, j = 1..(\tilde{k}+1)(n+1) - 1$. Therefore, the quadrature identity (27) is satisfied if the following system of $2(\tilde{k}+2)(n+1) - 1$ linear equations

$$V_j - U_j = 0, \quad \bar{V}_j - \bar{U}_j = 0, \quad j = 0..(\tilde{k}+2)(n+1) - 2 \tag{30}$$

for $2(\tilde{k}+1)(n+1)$ unknowns

$$Q, \bar{Q}, Q_j, \bar{Q}_j, \quad j = 1..(\tilde{k}+1)(n+1) - 1 \tag{31}$$

has solutions.

Note that the condition $\bar{Q} = Q$ is satisfied automatically, since, as easily seen from (27), (25), (12), the $j = 0$ subset of (30)

$$V_0 - U_0 = 0, \quad \bar{V}_0 - \bar{U}_0 = 0$$

are equations for $Q, \bar{Q}$ that have the same form for any $n \geq 0$ in (22). They have real solution

$$Q = \tilde{Q} = r^2 + \sum_{i=1}^{\tilde{k}} (i+1) u_i \bar{u}_i$$

The number of equations in (30) exceeds the number of unknowns (31) by $2n$ and the system of equations (30) is overdermined for the non-homogeneous medium problem $n > 0$.

For instance, for the circular domain $\tilde{k} = 0$

$$z(w) = z_1 + rw$$

in a medium with permeability $1/x^2$ (that has been considered in Section 3), system (30) consists of six equations

$$\begin{aligned}
&Q - r^2 = 0, \quad \bar{Q} - r^2 = 0, \\
&2Qx_1 - Q_1 + \bar{Q}_1 - 2x_1r^2 = 0, \quad 2\bar{Q}x_1 - \bar{Q}_1 + Q_1 - 2x_1r^2 = 0, \\
&4Q_1x_1 - r^4 = 0, \quad 4\bar{Q}_1x_1 - r^4 = 0
\end{aligned}$$

for four unknowns $Q, \bar{Q}, Q_1, \bar{Q}_1$. It has the following solution (already demonstrated in Section 3)

$$Q = \bar{Q} = r^2, \quad Q_1 = \bar{Q}_1 = r^4/4x_1$$

Returning to the general case, we are going to show that not all equations in (30) are independent, the system is compatible and has a unique solution, which proves our main result presented in Section 3.

To prove the compatibility, we introduce the basis

$$\phi_j(z,\bar{z}) = c_jT_n[(z-z_1)^{j+n}], \quad \bar{\phi}_j(\bar{z},z) = c_jT_n[(\bar{z}-\bar{z}_1)^{j+n}], \quad j = 0,1,2,... \tag{32}$$

where $c_j = j!/(j+n)!x_1^n$, in the space of solutions $\phi(z,\bar{z})$ of $L_n[\phi] = 0$ that are regular in neighborhood of $z = z_1$, and then show that the quadrature identity (13) holds for any element of this basis.

Indeed, according to (22), (32) is a set of solutions of $L_n[\phi] = 0$ that continuously tends to the basis of functions analytic in a neighborhood of $z = z_1$

$$\phi_j(z,\bar{z}) \to (z-z_1)^j, \quad j = 0,1,2,..., \quad x_1 \to \infty$$

as the position of the source $z_1 = x_1 + iy_1$ goes to infinity. On the other hand, $L_n[\phi(z,\bar{z})] = 0, z \in \Omega$ continuously tends to the Laplace equation when region Ω is moved to infinity. Therefore, set (32) is homotopically equivalent to $(z-z_1)^j, (\bar{z}-\bar{z}_1)^j, j = 0,1,2,..$ under a continuous deformation of the Laplace operator and thus contains a basis of solutions of $L_n[\phi] = 0$ that are regular in a neighborhood of $z = z_1$.

It then follows from (28) and (29) that the quadrature identity holds if

$$(V_j - U_j)\left(\frac{d^jf(z)}{dz^j}\right)_{z=z_1} = 0, \quad (\bar{V}_j - \bar{U}_j)\left(\frac{d^jf(\bar{z})}{d\bar{z}^j}\right)_{z=z_1} = 0, \quad j = 0..(\tilde{k}+2)(n+1)-2 \tag{33}$$

for

$$f(z) \in \{(z-z_1)^j, j \geq n\}$$

Since $j \geq n$, the left hand sides of the first n equations in (33) vanish, and there remains $2(\tilde{k}+1)(n+1)$ independent equations and the equal number of unknowns (31).

We now have to prove the compatibility of remaining equations. (30) is a non-homogenous system of linear equations for unknowns $Q, \bar{Q}, Q_j, \bar{Q}_j, j = 1..(\tilde{k}+1)(n+1)-1$, that fixes dependence of these unknowns on parameters $z_1, \bar{z}_1, r, u_1, ..., u_{\tilde{k}},\ \bar{u}_1, ..., \bar{u}_{\tilde{k}}$. The system is compatible if its homogenous part does not have nontrivial solutions. Let us suppose that it does. Remind that the homogenous part of the system has been obtained by action of the operator $\hat{Q}$ to an arbitrary solution of $L_n[\phi] = 0$ at point $z = z_1$. So, if the homogeneous part of the system had nontrivial solutions, then operator $\hat{Q}$ would annihilate any solution ϕ of $L_n[\phi] = 0$ at $z = z_1$, i.e.

$$\hat{Q} T_n[f(z)] = 0 \quad \text{at} \quad z = z_1, \tag{34}$$

where $f(z)$ is any analytic in Ω function. If the above were true, then changing continuously the position $z = z_1$ of the source, we could construct such operator $\hat{Q}$, with coefficients depending on z, that $\hat{Q} T_n[f(z)] = 0$ in some region of the plane for an arbitrary $f(z)$. But this is evidently impossible, since the highest symbols of $\hat{Q}$ and T_n contain pure derivatives in z, so does their composition.

Therefore (30) has a unique solution and quadrature identity holds in the polynomial algebraic domains, that leads to our main result (see Section 3).

Similar result can be analogously proved for the general system with permeability given by (18).

6 Generalization to Higher Dimensions

Our study can be extended to PDEs in more than two dimensions, since the derivation of the conservation laws in Section 2 is not restricted to the two-dimensional flows.

For instance, if we take solution $\phi(\xi)$ of $\nabla \xi_1^{-2} \nabla \phi = 0$, where $\zeta := (\xi_1, .., \xi_d)$ and $\nabla := \left(\frac{\partial}{\partial \xi_1}, .., \frac{\partial}{\partial \xi_d} \right)$, then the following quadrature identity holds

$$\int_{|\xi - \xi'| < r} \phi(\xi) d\xi_1 .. d\xi_d = v_d \left(\phi(\xi) + \frac{r^2}{(d+2)\xi_1} \frac{\partial \phi(\xi)}{\partial \xi_1} \right)_{\xi = \xi'}$$

in the d-dimensional ball of radius r with center at $\xi = \xi'$, where $v_d = \int_{|\xi - \xi'| < r} d\xi_1 .. d\xi_d$ is volume of the ball. Since the Hadamard expansion of the fundamental solution of the Calogero-Moser type operators truncates (see e.g. [3], [4]), the pressure distribution in the corresponding free-boundary flow can be written down explicitly for any d (e.g. we have used technique of [3] for derivation of (17)).

7 Conclusion, Main Conjecture

In this article we have found examples of non-homogeneous porous medium flows, driven by a multipole source at a fixed point, whose boundaries obey

the same dynamics as those of the homogeneous-medium flows also driven by a multipole source located at the same point. Namely, the medium permeability is a homogeneous rational function of x, y, and an invariant of the dihedral group. The multipole fluxes of the non-homogeneous medium problem must be fixed functions of those of the homogeneous medium one. The related variable-coefficient elliptic PDEs for the pressure distribution are of the integer multiplicity Calogero-Moser type and the quadrature identities for solutions of such equations hold in polynomial algebraic domains.

The above result has been derived using technique of intertwining operators. The fact that coefficients of these operators are polynomials in $z, \bar{z}$ is essential for algebraicity of the corresponding quadrature domains. This result can be extended in the following way

According to works on algebraic integrability [4], it is likely that the second-order elliptic operators related to the Coxeter root systems as well as their special deformations exhaust all possible operators that can be related to the Laplace operators through polynomial intertwining operators. These are elliptic operators for which a polynomial Baker-Akhieser function exists. We call all such operators the Algebraic Calogero-Moser operators. For instance, the algebraic Calogero-Moser equations [4] related to the porous medium problems (8) with the permeability

$$\kappa = x^{-2m}\left((2m+1)y^2 - x^2\right)^{-2}, \quad \eta = 1,$$

where $m = 2, 3, 4, ...$, are simplest examples that extend the main result of the present paper to non-Coxeter arrangements of mirrors.

More generally, the problem of classification of all such algebraic Calogero-Moser systems in two dimensions is as follows [1], [2], [4]: Find a strictly increasing sequence of integer positive numbers $0 \leq k_1 < k_2 < .. < k_n$ and a sequence $\omega_1, ..\omega_n$ of complex parameters ("phases"), such that the ratio of Wronskians

$$\zeta(x, y) = \rho^{k_n} \frac{W[\sin(\theta_1), .. \sin(\theta_{n-1}), \sin(\theta_n)]}{W[\sin(\theta_1), .. \sin(\theta_{n-1})]}, \quad \theta_j = k_j\theta + \omega_j, \quad z = \rho e^{i\theta}$$

is a polynomial in x, y. The corresponding intertwining operator is of the nth order and a polynomial in x, y that can be rewritten in the form of a Wronskian

$$T[f] = \rho^{k_n} \frac{W[\sin(\theta_1), ... \sin(\theta_n), f]}{W[\sin(\theta_1), ... \sin(\theta_{n-1})]}$$

with (24) being a special case. The results of the present paper (i.e. algebraicity of domains for PDEs with $\kappa = 1/\zeta(x, y)^2)$ also hold for the deformed systems.

In conclusion, one may pose the classification problem: *Find complete list of all PDEs whose solutions satisfy quadrature identities in polynomial algebraic domains*. In view of the above it is reasonable to expect that it has the following solution:

Conjecture: *The algebraic Calogero-Moser equations exhaust, up to a gauge equivalence, all possible second order elliptic PDEs whose solutions satisfy quadrature identities in polynomial algebraic domains*

As mentioned in Section 6, our study can be extended to PDEs in more than two dimensions. Although, we cannot use conformal maps to parametrize algebraic domains in higher dimensions, we can still pose the classification problem for domains with spherical boundaries, looking for a complete list of elliptic equations for which generalized mean value theorem holds. More precisely, one can look for equations in d-dimensions whose solutions possess the following property: *Integral of an arbitrary solution taken over an arbitrary d-dimensional ball, equals a linear combination (with coefficients depending on the ball radius and position) of the value of the solution and those of a finite number of its derivatives at the ball center.*

Extending our conjecture to higher dimensions we may expect that algebraic Calogero-Moser equations complete the above list. Note, that our classification problem in $d > 2$ seems to be equivalent to a restricted classical Hadamard problem for irreducible Huygens' operators that non-trivially depend on more than two variables. As in our case, all known examples of such Huygens' operators are related to algebraic Calogero-Moser systems [4].

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